LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **APRIL 2015**

MT 2811 - MEASURE THEORY AND INTEGRATION

Date : 18/04/2015 Dept. No. Max. : 100 Marks

Time : 01:00-04:00

Answer **ALL** questions.

01. (a) Prove that every interval is measurable . **(5)**

(OR)

(b) Show that, for any set A and any $\varepsilon > 0$, there is an open set G containing A such that $m^*(G) \leq m^*(A) + \varepsilon$. **(5)**

(c) (i) Let M be a collection of all Lebesgue measurable sets then prove that M is a σ-algebra in $\Re$.

(ii) Let c be any real number and f and g be real valued measurable functions defined on the same measurable set E. Then prove that $f + c$, $cf, f + g$ and fg are also measurable. **(8+7)**

OR

(d) Prove that the following statements regarding the set E are equivalent:

(i) E is measurable

(ii) $\forall \in > 0,$ there exists an open set $O \supseteq E$ such that $m^*(O - E) \leq \epsilon$.

(iii) $\exists G$, a G_δ –set, $G \supseteq E$ such that $m^*(G - E) = 0$.

(iv) $\forall \in > 0$, $\exists F$, a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \epsilon$

(v) $\exists F$, a F_σ –set, $F \subseteq E$ such that $m^*(E - F) = 0$. **(15)**

02. (a) Show that $\int_0^1 \frac{x^{\frac{1}{3}}}{1-x} log\frac{1}{x} dx = 9\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}$. **(5)**

OR

(b) If φ is a simple function then prove that $\int_{A\cup B} \emptyset dx = \int_A \emptyset dx + \int_B \emptyset dx$, for any disjoint measurable sets A and B and $\int a\emptyset dx = a\int \emptyset dx$, if a > 0. **(5)**

(c) Prove that if f is Riemann integrable and bounded over the finite interval $[a, b]$ then f is integrable and $R\int_a^b f dx = \int_a^b f dx$. **(15)**

OR

(d) State and prove Lebesgue monotone Convergence theorem. **(15)**

03. (a) (i) Define a ring and σ-ring. Prove that every algebra is a ring and every σ-algebra a σ-ring. **(5)**

OR

(ii) Define a measure and σ-finite measure on a ring $\Re$. Show that if μ is a σ-finite measure on a ring $\Re$, then the extension $\bar{\mu}$ of μ to S^*, the class of μ^*-measurable sets is σ-finite. **(5)**

(b) (i) Let $\{A_i\}$ be a sequence of sets in a ring R then prove that there is a sequence $\{B_i\}$ of disjoint sets of R such that $B_i \subseteq A_i$ for each i and $\bigcup_{i=1}^{N} A_i = \bigcup_{i=1}^{N} B_i$ for each N so that $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$. Also show that $\mathcal{H}(\mathrm{R}) = [\mathrm{E}: \mathrm{E} \subseteq \bigcup_{n=1}^{\infty} \mathrm{E}_n, \mathrm{E}_n \in \mathrm{R}]$. **(15)**

OR

(ii) Define a complete measure. Let μ^* be an outer measure on H(R) and let S^* denote the class of μ^* - measurable sets. Prove that S^* is a σ - ring and μ^* restricted to S^* is a complete measure **(15)**

04. (a) (i) Define a convex function and prove that for a convex function ψ on (a,b) such that $a < s < t < u < b$, then $\psi(s,t) \le \psi(s,u) \le \psi(t,u)$. **(5)**

OR

(ii) State and prove Minkowski Inequality. **(5)**

(b) (i) State and prove Jensen's inequality. When does equality occur?

(ii) Let $a > 0, b > 0, 1/p + 1/q = 1$ where $p > 1$and $q > 1$. Show that $a^{\frac{1}{p}} b^{\frac{1}{q}} \le \frac{a}{p} + \frac{b}{q}$.

(9+6)

OR

(c) Let $\{f_n\}$ be a sequence of measurable functions which is fundamental in measure. Then prove that there exists a measurable function f such that $f_n \to f$ in measure.

(d) Let ψ be a function on (a,b). Then prove that ψ is convex on (a,b) if and only if for each x and y such that $a < x < y < b$, the graph of ψ on (a,x) and (y,b) does not lie below the line through points in (x,ψ(x)) and (y, ψ(y)). **(10+5)**

05. (a) Define signed measure and show that a countable union of positive sets with respect to a signed measure υ is a positive set. **(5)**

OR

(b) Let υ be a signed measure and let μ, λ be measure on [X, S] such that μ, λ, v are σ - finite, $v \ll \mu$, $\mu \ll \lambda$ then prove that $\frac{dv}{d\lambda} = \frac{dv}{d\mu}\frac{d\mu}{d\lambda}[\lambda]$. **(5)**

(c) State and prove Lebesgue decomposition theorem. **(15)**

OR

(d) State and prove Radon-Nikodym theorem **(15)**

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